Low Reynolds Number Swimming

Joe Dinius
.
jdinius@math.arizona.edu

http://math.arizona.edu/~jdinius

Program in Applied Mathematics

University of Arizona

Problem Description

Consider the Naviér-Stokes equation for incompressible fluids

(1) $$\nabla p = \mu \nabla^2 u - \rho\Big(\frac{\partial u}{\partial t} + (u \cdot \nabla) u\Big) + f,$$

Through the introduction of the the Reynolds number, $R = \frac{\rho L U}{\mu}$, (1) can be made dimensionless. Because the characteristic length and velocity for the problem concerning motion of microorganisms are very small, the inertial contribution of (1) can be neglected.

Problem Description continued

The resulting equation governing low Reynold's number swimming is then

$$\nabla p = \mu \nabla^2 u + f \tag{2}$$

Because the ∇^2 operator acting on u is linear, the viscous force exerted on a fluid by a low Reynolds number swimmer scales directly with the stroke velocity applied by the swimmer. The consequence is that the presence of net motion is independent of rate.

Reversibility of Stokes Flow

Start video from G.I. Taylor

Stokeslets and Rotlets

The solution to (2) comes in the form of a linear superposition of the solutions [1]

$$\mathbf{U}_s(\mathbf{x};\mathbf{x}_0,\mathbf{f}_0) = \frac{\mathbf{f}_0}{8\pi r} + \frac{[\mathbf{f}_0\cdot(\mathbf{x}-\mathbf{x}_0)](\mathbf{x}-\mathbf{x}_0)}{8\pi r^3}$$

(Stokeslet)

$$\mathbf{U}_r(\mathbf{x};\mathbf{x}_0,\mathbf{L}_0) = \frac{\mathbf{L}_0\times(\mathbf{x}-\mathbf{x}_0)}{8\pi r^3}$$ (Rotlet).

Regularization

These solutions have singularities, therefore a regularization needs to be introduced. These regularized Stokeslets and Rotlets solve (2) and allow us to evaluate flow at all points. Introduce the cutoff function

$$\phi_\delta(\mathbf{x}) = \frac{15\delta^4}{8\pi(r^2+\delta^2)^{7/2}} \tag{3}$$

Convolving the Stokeslets and Rotlets with ϕ_δ gives the *regularized* Stokeslets and Rotlets

Regularization continued

$$\mathbf{U}_{\delta,s}(\mathbf{x};\mathbf{x}_0,\mathbf{f}_0) = \frac{\mathbf{f}_0(r^2+2\delta^2)}{8\pi(r^2+\delta^2)^{3/2}} + \frac{[\mathbf{f}_0\cdot(\mathbf{x}-\mathbf{x}_0)](\mathbf{x}-\mathbf{x}_0)}{8\pi(r^2+\delta^2)^{3/2}} \text{ (Stokeslet)}$$

$$\mathbf{U}_{\delta,r}(\mathbf{x};\mathbf{x}_0,\mathbf{L}_0) = \frac{(2r^2+5\delta^2)}{16\pi(r^2+\delta^2)^{5/2}}[\mathbf{L}_0\times(\mathbf{x}-\mathbf{x}_0)]$$

(Rotlet)

The error from regularization is $\mathcal{O}(\delta)+\mathcal{O}(\Delta s^2/\delta^3)$ near the body; $\mathcal{O}(\delta^2)+\mathcal{O}(\Delta s^2/\delta^3)$ away from it.

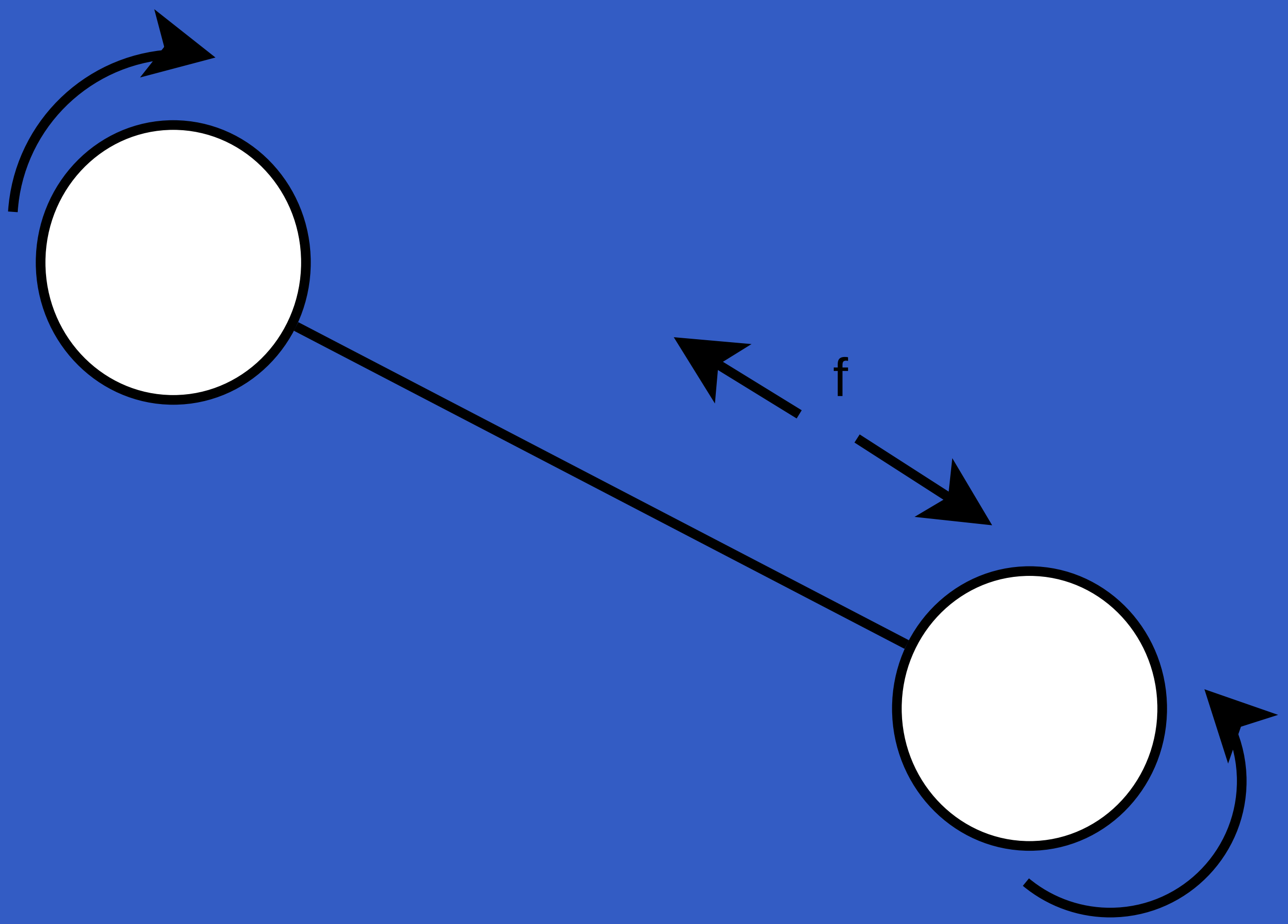
Solution of (2)

The solution at each point k along the discretization of swimmer length, through superposition of Stokeslets and Rotlets, can be represented by the simple first-order ODE

$$\mathbf{U}(\mathbf{x}_k) = \frac{d\mathbf{x}_k}{dt} = \sum_{i=1}^{N_r} \mathbf{U}_{\delta,r}(\mathbf{x}_k; \mathbf{x}_i, \mathbf{L}_i) \tag{4}$$

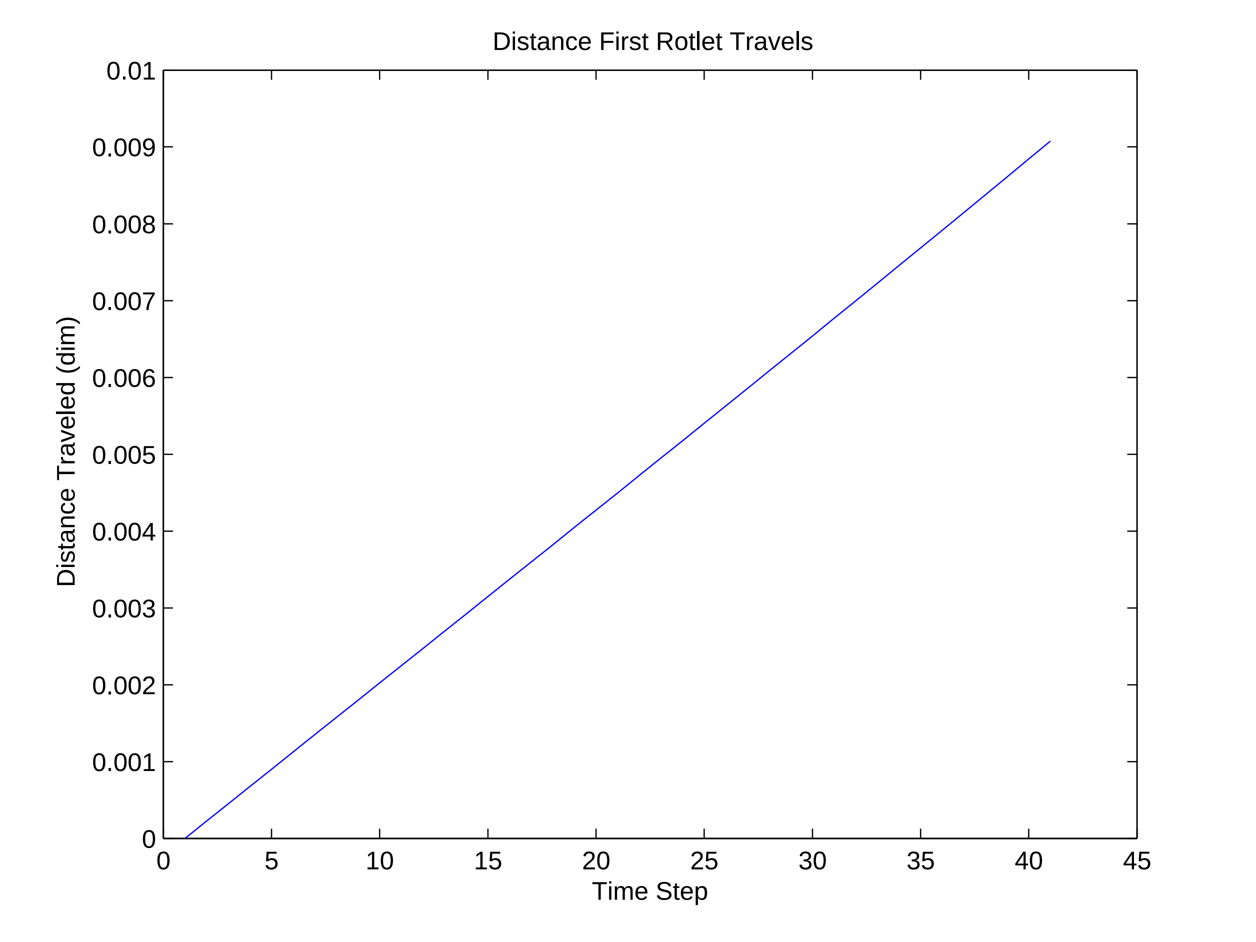
$$+ \sum_{j=1}^{N_s} \mathbf{U}_{\delta,s}(\mathbf{x}_k; \mathbf{x}_j, \mathbf{f}_j). \tag{5}$$

A Simple Swimmer

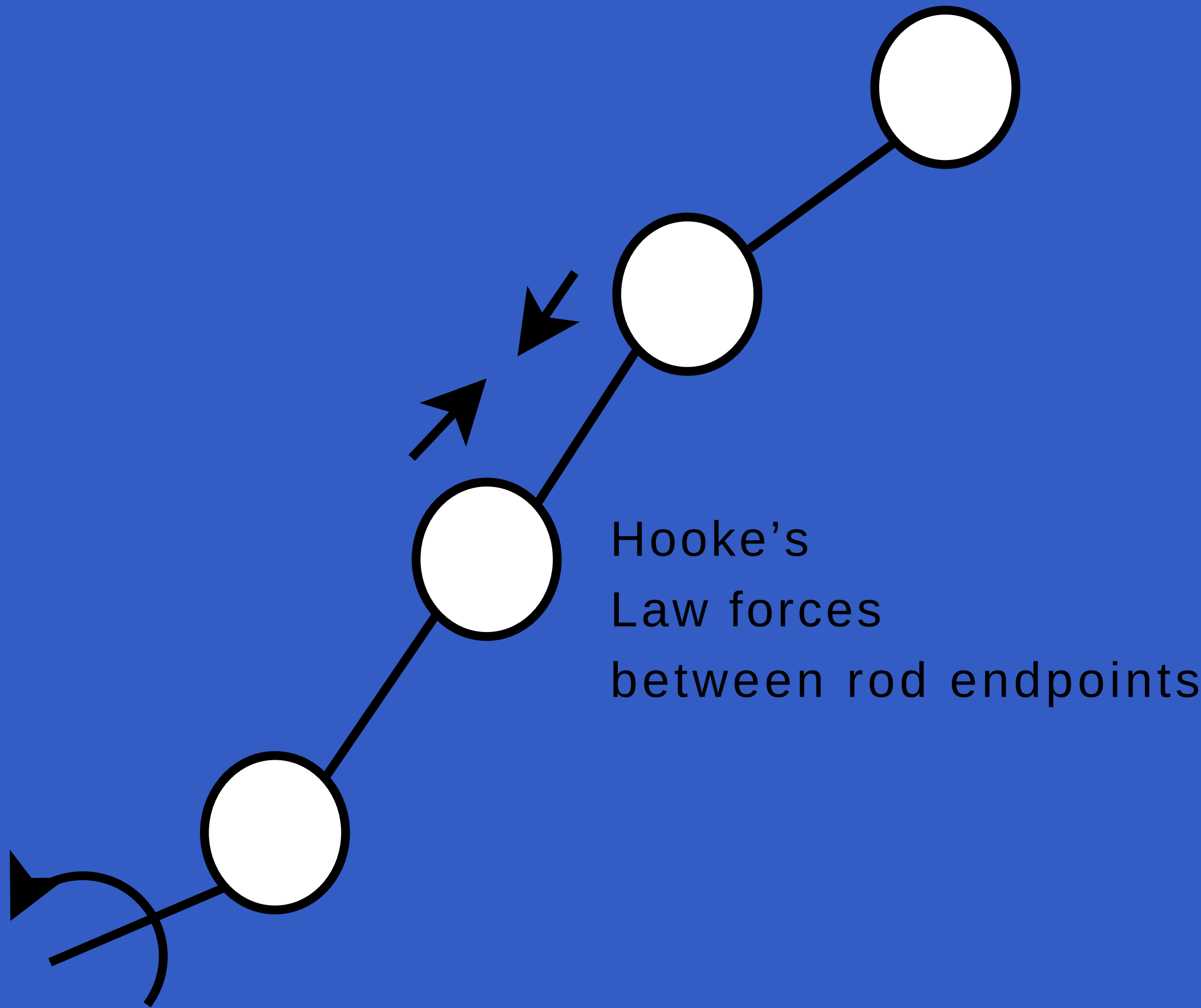


Movie of Simple Swimmer

Start external animation



Simple Swimmer- Rigid Rods



Motion of Rigid Rod Network

Consider different combinations resembling configuration on previous slide. Total rod length remains constant, while the number of Stokeslets is increased.

Distance Plot

Wrapup

- Stokes flows are reversible in time. Swimmers must have more than one degree of freedom to induce net motion.
- Solutions of Stokes flow for incompressible fluids come are the computationally-convenient regularized Stokeslets and Rotlets.
- Simple swimmers are easy to construct. One such simple swimmer was presented with fluid flow at a slice in the yz-plane. Vector field shows fluid balancing forces exerted by the swimmer.

References

[1] Flores, H., Lobaton, E., Méndez-Diez, S., Tlupova, S, Cortez, R., 2005. *A study of bacterial flagellar bundling*. Bulletin of Mathematical Biology. 67, 137-168.

[2] Purcell, E.M., 1977. *Life at Low Reynolds Number*. American Journal of Physics. 45, 3-11.

G.I. Taylor video available at

http://www.physics.nyu.edu/pine/research/hydroreverse.html